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Non-vanishing elements in finite groups

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Abstract

We show that if A is an elementary abelian normal p -subgroup of a finite group G and P is a Sylow p -subgroup of G , then no irreducible character of G vanish on any element of $Z(P) \cap A$.

Let G be a finite group. A well-known result of W. Burnside asserts that for each non-linear irreducible character χ of G , then there is an element $g \in G$ such that $\chi(g) = 0$. It is a natural question to determine elements of G which vanish for some irreducible character. Equivalently, we want to determine elements which do not vanish on any irreducible characters of G . Such an element is called a *non-vanishing* element. Nontrivial non-vanishing elements are not common. If G has a p -block of defect zero, then the order of confining element is prime to p . For example, none of simple group of Lie type has nontrivial non-vanishing elements.

Recently, I. M. Isaacs, G. Navarro and T. R. Wolf [1] have shown that if G is solvable, then the images of non-vanishing elements in $G/F(G)$ are 2-elements. They have also shown that if G has a normal Sylow p -subgroup P , then all elements of $Z(P)$ are non-vanishing in G as an easy result. We slightly extend this result.

Theorem 1 *If a finite group G possesses a nontrivial elementary abelian normal p -subgroup A and P is a Sylow p -subgroup of G , then all elements of $Z(P) \cap A$ are non-vanishing in G .*

Proof Let χ be an arbitrary irreducible character of G . Then we have

$$\chi|_A = e(\zeta_1 + \zeta_2 + \cdots + \zeta_r).$$

with some integer e , where $\chi|_A$ is the restriction of χ to A and $\{\zeta_1, \zeta_2, \dots, \zeta_r\} \subset \text{Irr}(A)$ is the set of all distinct conjugate characters for some $\zeta = \zeta_1 \in \text{Irr}(A)$ under the action of G . Let $H = \{g \in G | \zeta^g = \zeta\}$ be the inertia subgroup of ζ , where we define $\zeta^g(x) = \zeta(gxg^{-1})$. We have $r = |G : H|$.

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Now, by way of contradiction, suppose $\chi(x) = 0$ for $x \in Z(P) \cap A$. Note that $\deg(\zeta_j) = 1$ and the character values $\zeta_j(x)$ are p -th roots of 1. Pick a primitive p -root of unity ξ . It is well known that any $p - 1$ elements subset of $\{1, \xi, \dots, \xi^{p-1}\}$ is linearly independent over \mathbb{Q} and $1 + \xi + \dots + \xi^{p-1} = 0$. Then, in the expression :

$$\chi(x) = e(\zeta_1(x) + \zeta_2(x) + \dots + \zeta_r(x))$$

all p roots of $X^p - 1 = 0$ must appear with the same multiplicity $\frac{r}{p} = p^{m-1}q$. In particular, r is divisible by p .

We next consider the action of P on the set $\{\zeta^g | g \in H \setminus G\} = \{\zeta_1, \zeta_2, \dots, \zeta_r\}$. For each $g \in H \setminus G$, the size of the P -orbit of ζ^g is

$$|P : P \cap H^g| = |G : P \cap H^g|_p = (|G : H^g| |H^g : P \cap H^g|)_p = |G : H|_p |H^g : P \cap H^g|_p,$$

which is divisible by $|G : H|_p$, where $|S|_p$ denotes the highest power of the prime p that divides $|S|$.

Finally we consider the value $\zeta^g(x)$ on each orbit of P . That $x \in Z(P)$ implies $\zeta^{gy}(x) = \zeta^g(yxy^{-1}) = \zeta^g(x)$ for all $y \in P$. This implies that on each P -orbit of $\{\zeta^g | g \in H \setminus G\}$, the element x takes on the same value $\zeta^g(x)$. Hence the multiplicity $\frac{r}{p}$ must be divisible by $|G : H|_p$, which is obviously impossible. This completes the proof.

As a corollary, we have

Corollary 2 *If $G \neq \{1\}$ is solvable, then G possesses a nontrivial non-vanishing element.*

Proof Let A be a minimal normal subgroup of G . Then A is an elementary abelian p -group for some prime p . Also, $A \subseteq P$, where P is any Sylow p -subgroup of G . Now, $A \triangleleft P$, $Z(P) \cap A > 1$ and every $x \in Z(P) \cap A$ is non-vanishing.

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References

- [1] M. Isaacs, G. Navarro and T. Wolf, Finite group elements where no irreducible character vanishes, J. Algebra 222 (1999) 413–423.